The Scots College

Year 12 Mathematics Extension 2

Trial Assessment

August 2006

General Instructions

- All questions are of equal value
- Working time - 3 hours
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Standard Integrals Table is attached

TOTAL MARKS: 120

WEIGHTING: 40 %

- **Start each question in a new booklet**

QUESTION 1 [15 MARKS]

(a) For the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$, find

(i) the eccentricity 1

(ii) the coordinates of the foci 1

(iii) the equations of the directrices 1

(iv) Sketch the ellipse 1

(b) The point $P\left(ct,\frac{c}{t}\right)$ lies on the rectangular hyperbola $xy=c^2$.

(i) Sketch the hyperbola and mark on it the point P where $t \neq 0$ 1

(ii) Derive the equation of the tangent at P 2

(iii) Prove the equation of the normal at P is given by $y=t^2x+\frac{c}{t}-ct^3$ 1

(iv) The tangent at P meets the line y = x at T. Find the co-ordinates of T. 1

(v) The normal at P meets the line y =x at N. Find the co-ordinates of N. 1

(vi) Prove that $OT\times ON=4c^2$ (O is the Origin) 2

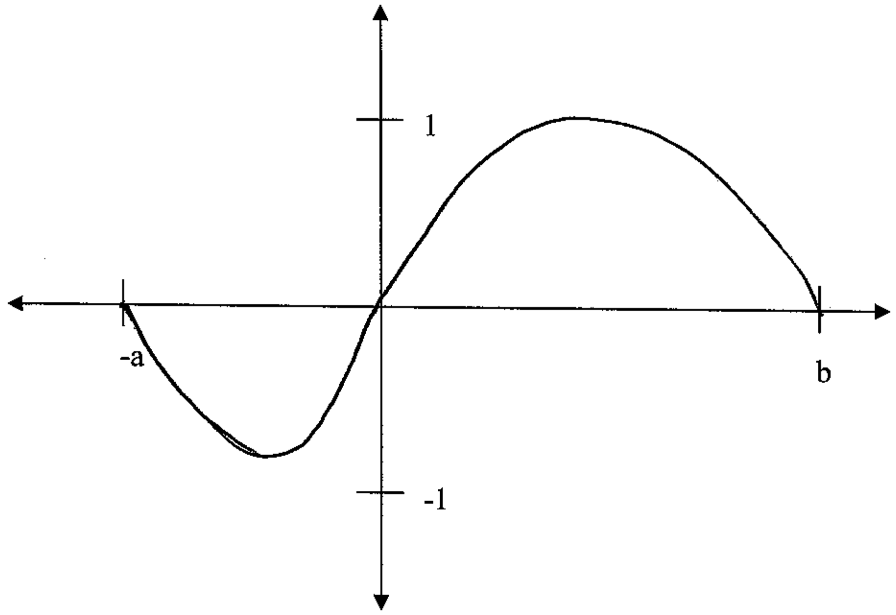
(c) The tangent to the hyperbola at a point P $(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes at Q and S.

If OQRS is a rectangle, where O is the Origin.

Find the locus of R. 3

QUESTION 2 **[15 MARKS]**

(a) The graph of the function f(x) is sketched below.
The domain of the function is $-a \geq x \geq b$ **Note b > a > 1**



On separate number planes sketch the graphs of:

(i) $y = f(-x)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = e^{f(x)}$ **2**

(iv) $y = f(x^2)$ **2**

(b) Given $G(x) = \dfrac{x^2-1}{x^2+1}$. On separate number planes sketch the graphs of:

(i) $y = G(x)$ **2**

(ii) $[G(x)]^2 = \dfrac{x^2-1}{x^2+1}$ **2**

(iv) $y = \dfrac{G(x)}{|G(x)|}$ **2**

(iii) $y = \dfrac{|x+1|(x-1)}{x^2+1}$ **2**

QUESTION 3 **[15 MARKS]**

(a) Given $z = 1 - i$, find:

(i) $\text{Im}\left(\frac{1}{z}\right)$ **2**

(ii) z^8 in the form $x + yi$ **2**

(iii) two values of w such that $w^2 = 3\bar{z} + i$ **3**

(b)

(i) Find the five fifth roots of unity **2**

(ii) If $w = cis\frac{2\pi}{5}$, show that $1 + w + w^2 + w^3 + w^4 = 0$ **1**

(iii) Show that $z_1 = w + w^4$ and $z_2 = w^2 + w^3$ are roots of the equation $z^2 + z - 1 = 0$ **2**

(c) In the Argand diagram the points A, B, C and D represent the complex numbers α, β, λ *and* δ respectively.

(i) Describe the point which represent $\frac{1}{2}(\alpha + \lambda)$ **1**

(ii) Deduce that if $\alpha + \lambda = \beta + \delta$ then ABCD is a parallelogram. **2**

QUESTION 4 **[15 MARKS]**

(a) Find the exact value of:

(i) $\int_0^{\pi} \sin^3 x \, dx$ **2**

(ii) $\int_1^2 \frac{(\log_e x)^2}{x} dx$ **2**

(iii) $\int_0^{\frac{1}{2}} \cos^{-1} x \, dx$ **2**

(b) Use the substitution $u = 2 + \cos\theta$ to show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2+\cos\theta} d\theta = 2 + 4\log_e\left(\tfrac{2}{3}\right)$$ **3**

(c) By using $\tan\left(\frac{x}{2}\right) = t$, integrate

$\int \sec x dx$ **2**

(d)

(i) Explain why $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **2**

(ii) Hence show that $\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx = \frac{\pi}{4}$ **2**

QUESTION 5 **[15 MARKS]**

(a) A polynomial function is $P(x) = x^5 + x^4 + 13x^3 + 13x^2 - 48x - 48$.
Factorise P(x) over the field of:

(i) real numbers **2**

(ii) imaginary numbers **1**

(b) Solve $2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$ if it has a root of multiplicity of 3. **3**

(c) Let $\alpha, \beta\, and\, \lambda$ be the roots of a cubic equation $x^3 + px^2 + q = 0$, where $p\ and\, q$ are real.
The equation $x^3 + ax^2 + bx + c = 0$ has the roots $\alpha^2, \beta^2\ and\ \lambda^2$.

Find $a, b\, and\, c$ in terms of $p\ and\ q$ **3**

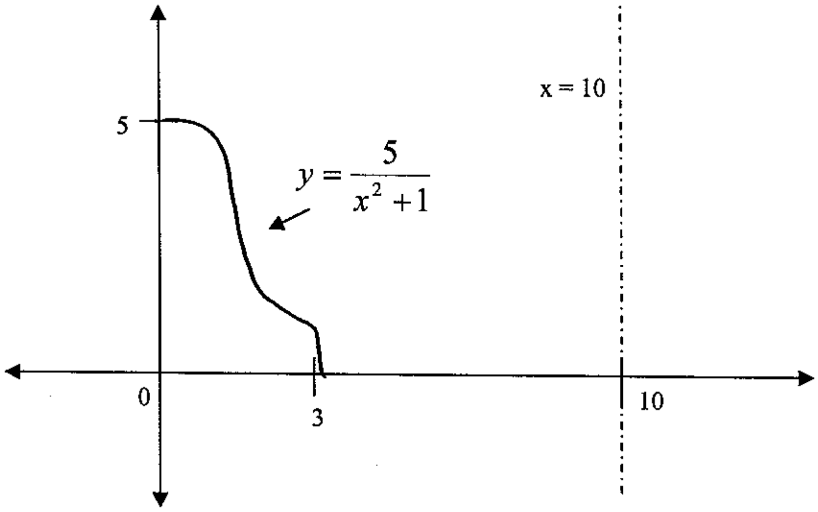
(d) A monic cubic polynomial, when divided by $x^2 + 4$ it leaves a remainder of $x + 8$.
When it is divided by x it leaves a remainder of -4.
Find the polynomial in expanded form **2**

(e) By using De Moivre's theorem, show that the expansion of
$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ **2**

Hence solve the equation $16x^4 - 20x^2 + 5 = 0$ **2**

QUESTION 6 **[15 MARKS]**

(a)



A circular flange is formed by rotating the region bounded by the curve $y = \frac{5}{x^2+1}$, the x axis and the lines x = 0 and x = 3, about the line x = 10.
(All measurements are in cm)

(i) Use the **method of cylindrical shells** to show the volume generated V cm^3 of the flange is given by $V = \int_0^3 \frac{(100-10x)\pi}{x^2+1}dx$. **2**

(ii) Hence find the volume of the flange correct to the nearest cm^3 **2**

(b) Use the **method of slices** to find the volume generated when the area bounded by $y = x^2 - 3x^4$ and the x-axis is rotated about the y-axis. Begin with a suitable sketch.

(i) Evaluate the volume. **4**

(ii) Give two reasons why using the **"method of cylindrical shells"** in this question would have been easier. **1**

(iii) Use the **method of cylindrical shells** to evaluate the volume **2**

(c) Show the area of an isosceles right angled triangle with hypotenuse h is given by $\frac{h^2}{4}$ **1**

The base of a solid is the region enclosed by the curve $y = 9 - x^2$ and the x-axis.
Each cross-section perpendicular to the y-axis is an isosceles right angled triangle with hypotenuse lying in the base.
Use integration to find the volume of the solid. **3**

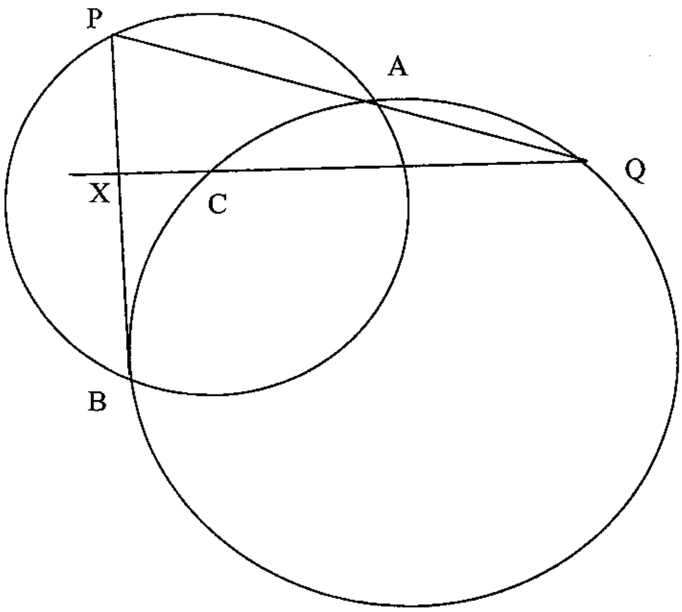
QUESTION 7 **[15 MARKS]**

(a) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx \quad n \geq 0$

(i) Show that $I_n = \dfrac{n-1}{n} I_{n-2}$ **2**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x dx$ **2**

(b)



The two circles intersect at A and B. the larger circle passes through the centre, C, of the smaller circle. P and Q are points on the circle such that PQ passes through A. QC is produced to meet PB at X

Make a neat copy of the diagram on your answer sheet. **Let** $\angle QAB = \theta$

(i) Show that $\angle BCX = 180° - \theta$ **2**

(ii) Prove that $\angle PXC = 90°$ **3**

(c)

(i) If a and b are positive real numbers prove $\dfrac{a+b}{2} \geq \sqrt{ab}$ **1**

(ii) Hence if a , b and c are positive real numbers prove $(a+c)(b+c)(b+a) \geq 8abc$ **1**

Hence if a, b and c are sides of a triangle, assuming that $0 < c \leq b \leq a$, prove that;

(iii) $(a+b-c)(b+c-a)(c+a-b) \leq abc$ **2**

(iv) $a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 3abc$ **2**

QUESTION 8 **[15 MARKS]**

A sequence U_n has the general term $\frac{1}{(n+1)(n+2)}$ for all integers $n \geq 1$

(i) Prove by Induction that $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{n}{2(n+2)}$ **3**

(ii) Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)}$. Let this be called S **2**

(iii) Evaluate $\int_1^{\infty} \frac{dx}{(x+1)(x+2)}$. Let this be called I **3**

(iv) Sketch the graph of the function $y = \frac{1}{(x+1)(x+2)}$ *for* $x \geq 0$ **2**

(v) Using the Trapeziodal Rule, with equal strips of unit width, find and approximate area under the the function $y = \frac{1}{(x+1)(x+2)}$ *for* $x \geq 0$. Let this be called T **3**

(vi) Using graphic means, explain the relationship between S, I and T, in terms of their size. **2**

END OF PAPER

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a \neq 0$$

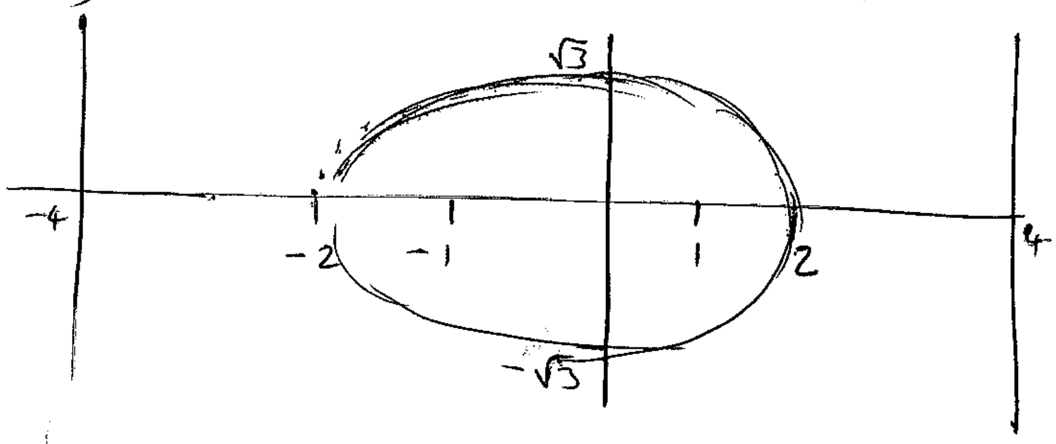
$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

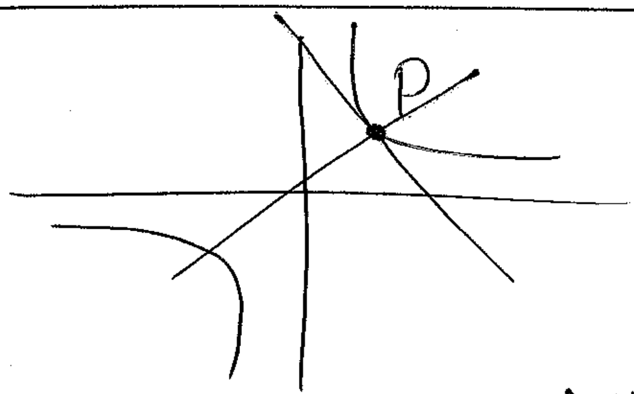
$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE : $\ln x \equiv \log_e x, \quad x > 0$

Qu 1 a) $e = \frac{1}{2}$, Foci $(\pm 1, 0)$, Directrix $x = \pm 4$



b)



$y = c^2/x$

$y' = -c^2/x^2$

$= \frac{-c^2}{(ct)^2} = -1/t^2$

$y - c/t = -1/t^2(x - ct)$

$\underline{x + yt^2 - 2ct = 0}$

$\searrow$ T

$\left(\frac{2ct}{(1+t^2)}, \frac{2ct}{(1+t^2)}\right)$

Normal $y' = t^2$

$y - c/t = t^2(x - ct)$

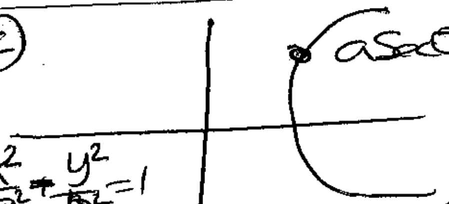
$\underline{y = t^2 x + c/t - ct^3}$

N $\left(\frac{c(1+t^2)}{t}, \frac{c(1+t^2)}{t}\right)$

ON $= \sqrt{2}\,\frac{c(1+t^2)}{t}$ OT $= \sqrt{2}\,\frac{2ct}{1+t^2}$

ON . OT $= 4c^2$. QED

c)



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$m = \frac{b\sec\theta}{a\tan\theta}$

Tangent $\frac{\sec\theta\, x}{a} - \frac{\tan\theta\, y}{b} = 1$

$x = 0$ $y = \frac{-b}{\tan\theta}$

$y = 0$ $x = a/\sec\theta$

$[\sec^2\theta - \tan^2\theta = 1]$

Locus $\underline{\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1}$

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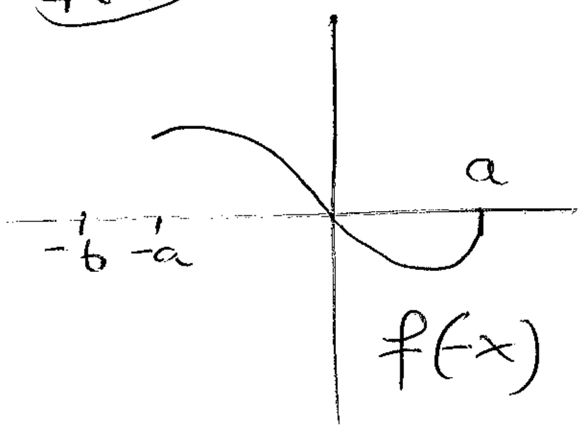
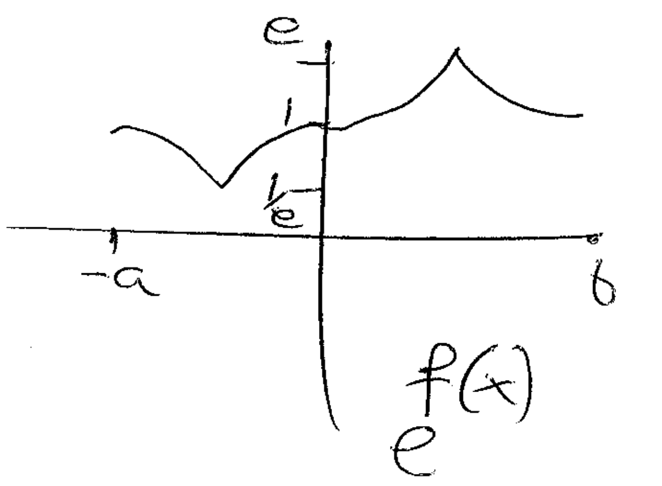
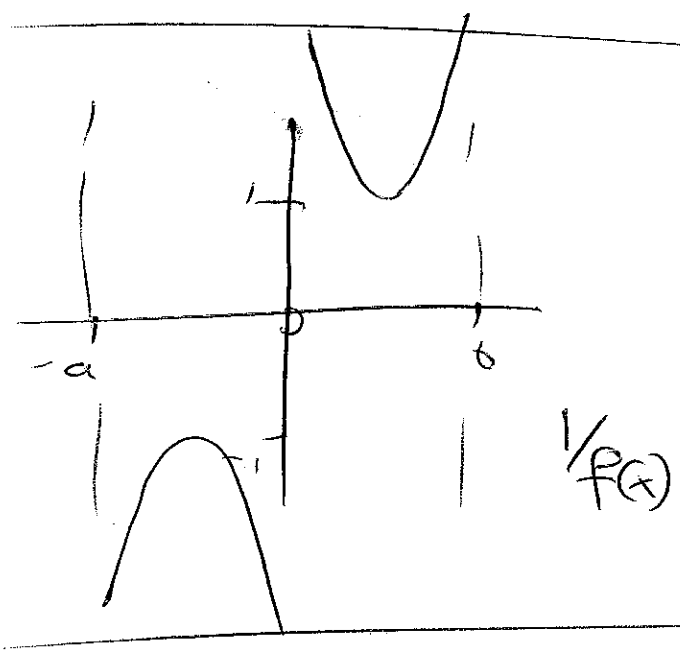
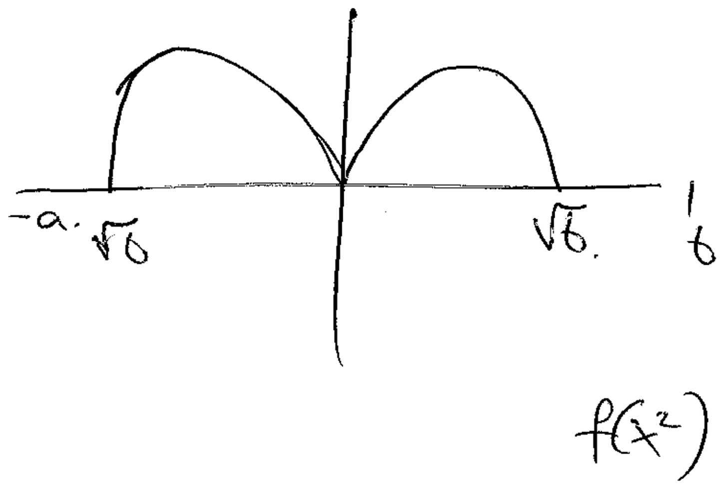
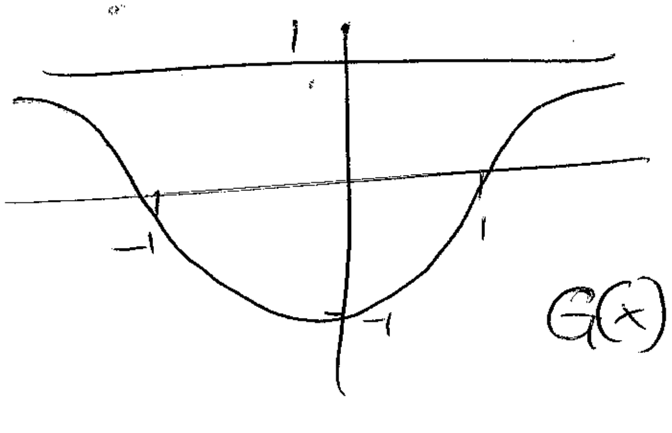
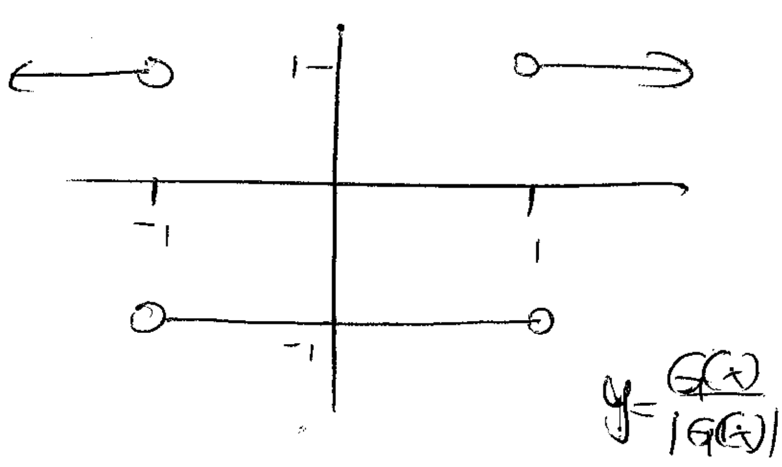

a
-b
-a
f(-x)

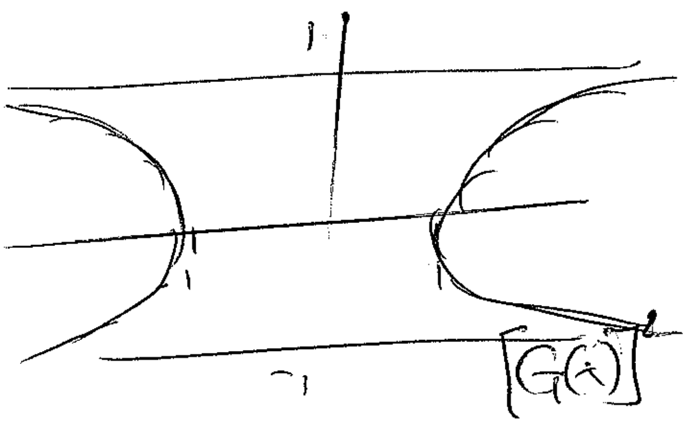

e
1
1/e
-a
b
f(x)
e

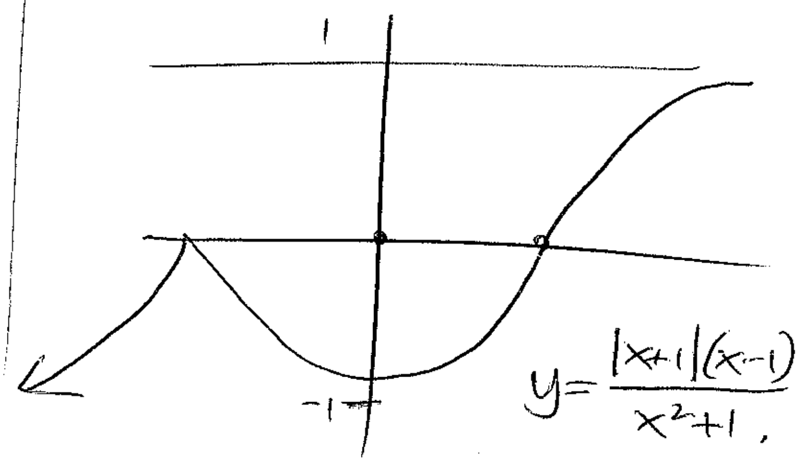

1
-a
b
-1
1/f(x)


-a
√b
√b
b
f(x²)


1
-1
1
-1
G(x)


1
-1
1
-1
y = G(x)/|G(x)|


1
1
1
-1
[G(x)]


1
-1
y = |x+1|(x-1)/(x²+1)

Qu 3

a) i) $Z = 1 - i$

$\frac{1}{Z} = \frac{1}{1-i}$

$= \frac{1}{1-i} \times \frac{1+i}{1+i}$

$= \frac{1}{2} + \frac{1}{2}i$

$\therefore \mathrm{Im}\left(\frac{1}{Z}\right) = \frac{1}{2}$

ii) $Z = \sqrt{2}\,\mathrm{cis}\,{-\frac{\pi}{4}}$

$Z^8 = \sqrt{2}^8\,\mathrm{cis}\left(-\frac{\pi}{4}\cdot 8\right)$

$= 16\,\mathrm{cis}(-2\pi)$

$= 16 + 0i$

iii) $w^2 = 3(1+i) + i$

$= 3 + 4i$

$\sqrt{3+4i} = x + yi$

$3 + 4i = (x+yi)^2 = x^2 - y^2 + 2xyi$

$x^2 - y^2 = 3 \quad 2xy = 4$

$x^4 + 3x^2 - 4 = 0$

$(x^2+4)(x^2-1) = 0$

Not Real

$y = \pm 1 \Rightarrow x = \pm 2$

$w = 2 + i \;, -2 - i$.

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b) cis 0

$\therefore w = \mathrm{cis}\,0$

$w_1 = \mathrm{cis}\,\frac{2\pi}{5}$

$w_2 = \mathrm{cis}\,\frac{4\pi}{5}$

$w_3 = \mathrm{cis}\,\frac{-2\pi}{5}$

$w_4 = \mathrm{cis}\left(\frac{-4\pi}{5}\right)$

ii) $w^5 = 1$

$w^5 - 1 = 0$

$(w-1)(w^4 + w^3 + w^2 + w + 1) = 0$

$= 0$.

iii) Sum of Roots $= -1$

Product of Roots $= -1$

$(w + w^2 + w^3 + w^4) = -1$

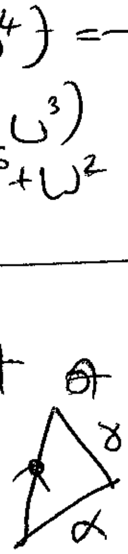
$(w + w^4)(w^2 + w^3)$

$w^3 + w^4 + w^6 + w^2 = -1$

w.

c)

i) Mid Point of Line



ii) $\frac{\alpha + \lambda}{2} = \frac{\beta + \gamma}{2}$ Mid Points at same point

QU4 $\int_0^{\pi} \sin^3 x\,dx$

$\int_0^{\pi} \sin x \sin^2 x\,dx$

$\int_0^{\pi} \sin x(1-\cos^2 x)\,dx$

$\int \sin x - \sin x\cos^2 x\,dx$

$\left[-\cos x - \frac{1}{3}\cos^3 x\right]_0^{\pi}$

$\underline{-4/3}$

$\int_1^2 \frac{(\log_e x)^2}{x}\,dx$

$\left(\frac{[\log_e x]^3}{3}\right)_1^2$

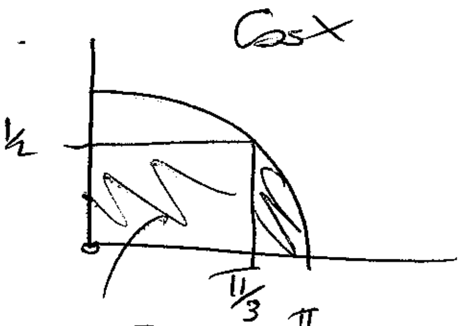
$\frac{1}{2}(\ln 2)^3$

$\int_0^{1/2} \cos^{-1} x\,dx$ → By Parts / Or $\cos x$.

$v = \cos^{-1} x \qquad du = 1$

$dv = \frac{-1}{\sqrt{1-x^2}} \qquad u = x$

⋮



$\frac{\pi}{6} + \int_{\pi/3}^{\pi/2} \cos x\,dx$

$\frac{\pi}{6} + [\sin x]_{\pi/3}^{\pi/2}$

$\frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}$

$\int_0^{\pi/2} \frac{\sin 2\theta}{2+\cos\theta}\,d\theta$

$u = 2 + \cos\theta$

$du = -\sin\theta\,d\theta$

$\cancel{d\theta} =$

$\int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{2+\cos\theta}\,d\theta$

$\int_3^2 \frac{2-u}{u}\,du$

$\int_3^2 \frac{2}{u} - 1\,du \rightarrow \left[2\ln u - u\right]_3^2$

⋮

QED

4 e)

$$\int \sec x \, dx = \int \frac{1}{\cos x} dx$$

$$\cos x = \frac{1+t^2}{1-t^2} \qquad d\cancel{t} = \tfrac{1}{2}\sec^2 \tfrac{x}{2} dx$$

$$= \tfrac{1}{2}(1+t^2)dx$$

$$dx = \frac{2dt}{1+t^2}$$

$$\int \frac{\cancel{1+t^2}}{1-t^2} \cdot \frac{2dt}{\cancel{1+t^2}}$$

Parti.. $\displaystyle \int \frac{1}{1-t} + \frac{1}{1+t} dt$

$$\log\frac{(1+t)}{(1-t)} + C \Rightarrow \log\frac{(1+\tan\frac{x}{2})}{(1-\tan\frac{x}{2})} + c$$

$$\int_0^a f(x)\,dx$$

let $\cancel{u} = a - x$

$\therefore$ Limits $a \Rightarrow 0$
$0 \Rightarrow a$

$$\frac{d\cancel{u}}{d\cancel{x}} = -1$$

$$\therefore -\int_a^0 f(u)\,du$$

$$= \int_0^a f(u)\,du$$

or by diagram.

$$I = \int_0^{\pi/2} \frac{\sin^m x}{\sin^m x + \cos^m x} dx$$

$$= \int_0^{\pi/2} \frac{\sin^m(\frac{\pi}{2}-x)}{\sin^m(\frac{\pi}{2}-x) + \cos^m(\frac{\pi}{2}-x)} dx$$

$$I = \int_0^{\pi/2} \frac{\cos^m(x)}{\cos^m(x) + \sin^m(x)} dx$$

$$2I = \int_0^{\pi/2} \frac{\sin^m x + \cos^m x}{}$$

$$2I = \int_0^{\pi/2} 1\,dx$$

$$= \left[x\right]_0^{\pi/2}$$

$$2I = \frac{\pi}{2} \Rightarrow I = \pi/4$$

5

a) $P(-1) = 0$

$x+1$ is a factor

$$x+1 \overline{\big)\quad} \quad x^4+13x^2-48$$

$(x+1)(x^4+13x^2-48)$

$(x+1)(x^2+16)(x^2-3)$

$\quad\quad\quad\quad I \quad\quad\quad R$

$(x+1)(x-\sqrt{3})(x+\sqrt{3})(x^2+16)$

$(x+1)(x-\sqrt{3})(x+\sqrt{3})(x-4i)(x+4i)$

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b) $2x^4+9x^3+6x^2-20x-24=0$

mult 3 $\therefore$ $f(x)=0$ $f'(x)=0$ $f''(x)=0$

$24x^2+54x+12=0$

$(4x+1)(x+2)$

$x=-1/4 \quad x=-2$

One is triple root & turn pos, one is turn point only

$P(-2)=0$

$(x+2)(x+2)(x+2)(ax+b)$

By inspecti $a=2,\ b=-3$

$\therefore$ Solve $x=-2,\ 3/2$

c) $P(x) = x^3 + px^2 + q = 0 \qquad \alpha, \beta, \lambda.$

For $\alpha^2, \beta^2, \lambda^2 \quad P(x^{1/2}) = (x^{1/2})^3 + p(x^{1/2})^2 + q$

$$x^{3/2} + px + q = 0$$

$$x^{3/2} = -px - q$$

$$(x^{3/2})^2 = (-px - q)^2$$

$$x^3 = p^2x^2 + 2pqx + q^2$$

$$x^3 - p^2x^2 - 2pqx - q^2$$

$$x^3 + ax^2 + bx + c.$$

$$a = -p^2 \quad b = -2pq \quad c = -q^2$$

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d) $P(x) = 1x^3 + ax^2 + bx + c$

$$P(x) = x^2 + 4\, Q(x) + x + 8$$

By deduction $Q(x) = (x + a)$

$$(x^2+4)(x+a) + x + 8$$

$$x^3 + ax^2 + 4x + x + 4a + 8$$

$$\therefore b = 5 \quad \text{and} \quad 4a + 8 = c$$

$$P(x) = x\,R(x) - 4$$

By deduct $R(x) = x^2 + ax + b$

$$x(x^2 + ax + b) - 4$$

$$x^3 + ax^2 + bx - 4.$$

$$\therefore c = 4$$

$$4a + 8 = c \Rightarrow a = -3$$

$$a = -3, \; b = 5, \; c = 4$$

e) $(\text{cis}\,\theta)^5 = 1^5(\cos 5\theta + i\sin 5\theta)$

By De Moivre's

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

$\therefore (\text{cis}\,\theta)^5 = \underline{c^5} + 5c^4 iS + \underline{10c^3 i^2 S^2} + 10c^2 i^3 S^3 + \underline{5c i^4 S^4} + S^5$

Real

$\cos(5\theta) = \cos^5\theta - 10c^3S^2 + 5cS^4$

$= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2$

$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$

$= 16c^5 - 20c^3 + 5c$

$\cos 5\theta = c(16c^4 - 20c^2 + 5)$

When does this equal 0 $\Rightarrow \cos 5\theta = 0$

$\therefore 5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}.$ [Repeats]

5 Solution $\Rightarrow$ 1 $\Rightarrow$ c

4 $\Rightarrow 16c^4 - 20c^2 + 5$

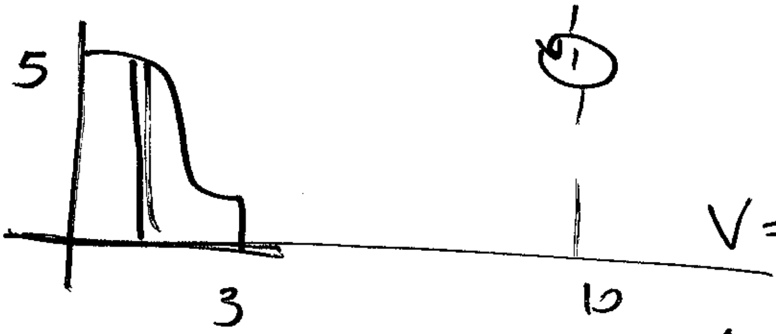
$\cos\frac{5\pi}{10} = \cos\frac{\pi}{2} = 0$ This is the c solution

(As a check $16x^4 - 20x^2 + 5 \neq 0$ when $x = \cos\frac{\pi}{2} = 0$.)

Solut $x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$

6

a)



Radius $= (10 - x)$

Height $= y$

$$V = 2\pi \int_0^3 (10-x) y \, dx$$

$$V = 2\pi \int_0^3 (10-x) \frac{5}{x^2+1} \, dx$$

$$V = \int_0^3 \frac{(100 - 10x)\pi}{x^2+1} \, dx$$

$$V = 10\pi \int_0^3 \frac{10 - x}{x^2+1} \, dx$$

$$= 10\pi \int_0^3 \frac{10}{x^2+1} - \frac{x}{x^2+1} \, dx$$

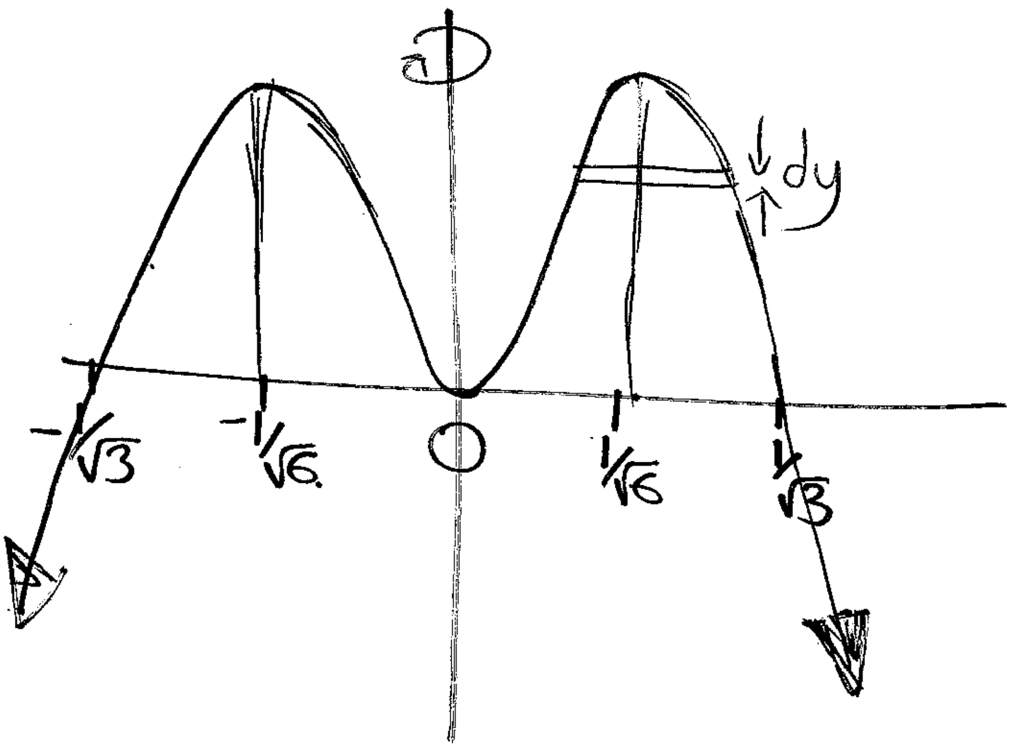
$$= 10\pi \left[10\tan^{-1} x - \tfrac{1}{2}\ln(x^2+1) \right]_0^3$$

$$= 10\pi \left[10\tan^{-1} 3 - \tfrac{1}{2}\ln 10 \right]$$

$\simeq$

nearest cm.

b)



$x^2(1 - 3x^2) = 0$

$x^2 = 0 \quad 3x^2 = 1$

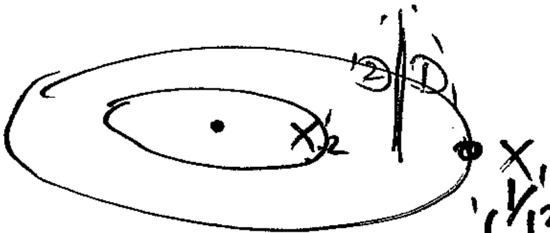
$x = 0 \quad x = \pm \frac{1}{\sqrt{3}}$

$y = x^2 - 3x^4$

$\frac{dy}{dx} = 2x - 12x^3$

$0 = 2(x)(1 - 6x^2)$

$x = 0 \quad x = \pm \frac{1}{\sqrt{6}}$.



$$V = \pi \int_0^{1/12} \left[x_1^2 - x_2^2 \right] dy$$

Split

① $$\pi \int_0^{1/12} x^2 - \left(\frac{1}{\sqrt{6}}\right)^2 dy$$ + ② $$\pi \int_0^{1/12} \left(\frac{1}{\sqrt{6}}\right)^2 - x^2 \, dy$$

Change to dx

$$\frac{dy}{dx} = 2x - 12x^3$$

$$= \pi \int_{1/\sqrt{3}}^{1/\sqrt{6}} \left(x^2 - \frac{1}{6}\right)\left(2x - 12x^3\right) dx + \pi \int_0^{1/\sqrt{6}} \left(\frac{1}{6}\right) - x^2 (2x - 12x^3)$$

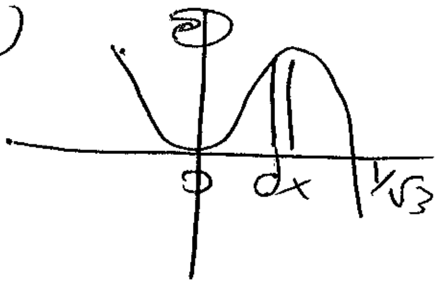
$$= \pi \int_{1/\sqrt{3}}^{1/\sqrt{6}} \left(4x^3 - 12x^5 - \frac{x}{3}\right) dx + \pi \int_0^{1/\sqrt{6}} \left(\frac{x}{3} - 4x^3 + 12x^5\right) dx$$

⋮

$$= \frac{\pi}{54} \, u^3$$

ii) Difficult to connect x_1 and x_2
Need to change to dx and hence limits
Much more complicated algebra

b) iii)



height = y
radius = x

$\therefore V = 2\pi\int_0^{1/\sqrt{3}} xy\,dx$

$= 2\pi\int_0^{1/\sqrt{3}} x(x^2-3x^4)\,dx$

$= 2\pi\int_0^{1/\sqrt{3}} x^3-3x^5\,dx$

$= 2\pi\left[\frac{x^4}{4}-\frac{x^6}{2}\right]_0^{1/\sqrt{3}}$

$= \pi/54 \; u^3$

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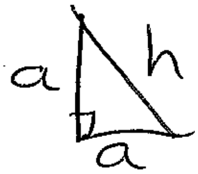
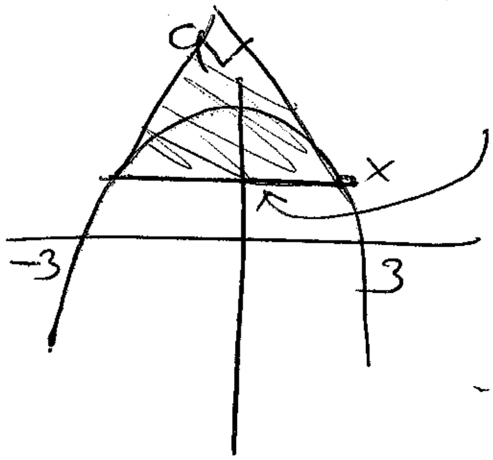
c)



$\left.\begin{array}{l} a^2+a^2=h^2 \\ \therefore 2a^2=h^2 \\ a^2=\frac{h^2}{2} \\ a=h/\sqrt{2} \end{array}\right\}$ Area $= \frac{1}{2}\times a\times a$

$= \frac{1}{2}\times\frac{h}{\sqrt{2}}\times h/\sqrt{2}$

$= \underline{\underline{h^2/4}}$



Hypotenuse = $2x = h$.

$\therefore$ Area $= \frac{h^2}{4} = \frac{(2x)^2}{4}$

$= x^2$.

$\therefore V = \int_0^9 x^2\,dy$

$= \int_0^9 (9-y)\,dy$

$= 9y - \frac{y^2}{2}$

$= \left[9y-\frac{y^2}{2}\right]_0^9$

$81-\frac{81}{2}$

$= 81/2 \; u^3$

⑦ $I_n = \int_0^{\pi/2} \cos^n x\,dx$

$= \int_0^{\pi/2} \cos x \cos^{n-1} x\,dx$

Integration by Parts.

$I_n = \left[\cancel{\sin x \cos^{n-1} x}\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x (n-1)\cos^{n-2} x - \sin x$

$= +(n-1)\int_0^{\pi/2} \sin^2 x \cos^{n-2} x\,dx$

$= (n-1)\int_0^{\pi/2} (1-\cos^2 x)\cos^{n-2} x\,dx$

$= (n-1)\left[\underbrace{\int_0^{\pi/2} \cos^{n-2} x\,dx}_{I_{(n-2)}} - \underbrace{\int_0^{\pi} \cos^n x\,dx}_{I_n}\right]$

$I_n = (n-1)I_{(n-2)} - (n-1)I_n$

$nI_n = (n-1)I_{n-2}$

$\therefore\ I_n = \frac{n-1}{n} I_{(n-2)}$

$I_4 = \frac{3}{4} I_2$

$= \frac{3}{4} \times \frac{1}{2} I_0$

$= \frac{3}{8}\int \cos^0 x\,dx$

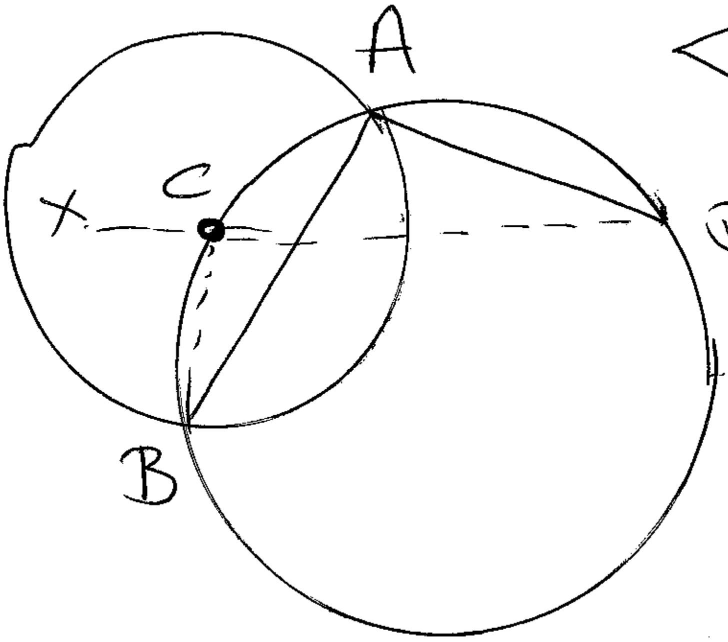
$= \frac{3}{8}\int 1\,dx$

$= \frac{3}{8}\left[x\right]_0^{\pi/2}$

$= \frac{3\pi}{16}.$

7b)

i)



$\angle PAB = \theta$

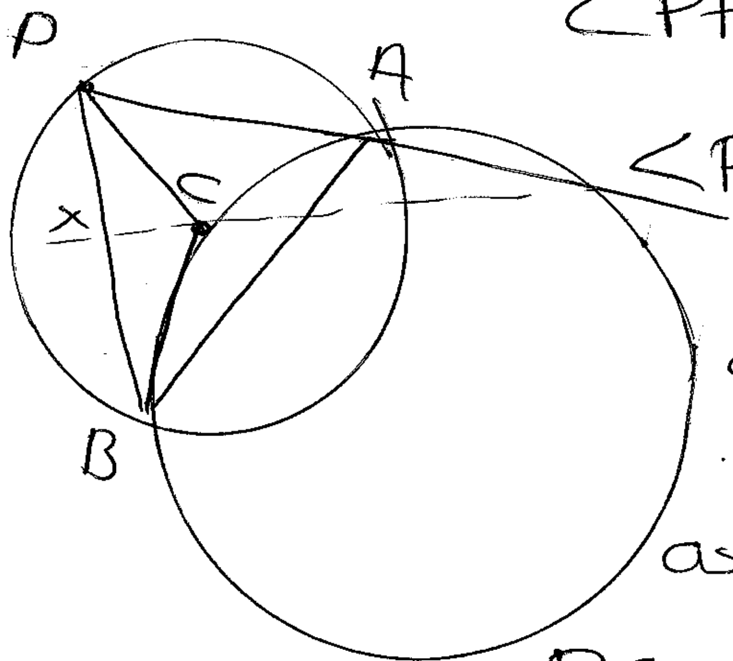
$\angle PCB = \theta$

(Angles on same arc)

$\therefore \angle BCX = 180 - \theta$

(Angles at a straight line)

ii)



$\angle PAB = 180 - \theta$

Angle at straight line.

$\angle PCB = 2\angle PAB$

Angles at centre = 2 × Angle at circum

$\angle BCX = 180 - \theta$

$\therefore \angle PCX = 180 - \theta$

as $PCB = 2(180 - \theta)$

$PC = BC$ Radii

XC Common

$\therefore \triangle PCX \equiv \triangle BCX$

$\therefore PB \perp XC$.

7 c) $(a-b)^2 \geqslant 0$

$$a^2 - 2ab + b^2 \geqslant 0$$

$$a^2 + b^2 \geqslant 2ab.$$

$$(a+b)^2 - 2ab \geqslant 2ab$$

$$(a+b)^2 \geqslant 4ab$$

$\sqrt{\ }$ as both +ve

$$a+b \geqslant 2\sqrt{ab}$$

$$\therefore \frac{a+b}{2} \geqslant \sqrt{ab}.$$

$a+b \geqslant 2\sqrt{ab} \quad a+c \geqslant 2\sqrt{ac} \quad b+c \geqslant 2\sqrt{ac}$

$$\therefore (a+b)(a+c)(b+c) \geqslant 2\sqrt{ab}\, 2\sqrt{ac}\, 2\sqrt{bc}$$

$$\geqslant 8abc$$

Let $A = b+c-a$, $B = a+c-b$, $C = a+b-c$

$(A+B)(A+C)(B+C) \geqslant 8ABC$ from above

but $A+B = 2c$ $A+C = 2b$ $B+C = 2a$

$$\therefore (b+c-a)(a+c-b)(a+b-c) = 8abc$$

Cancel 8 $\quad \therefore abc \geqslant (b+c-a)(a+c-b)(a+b-c)$

Given $[abc - a^2(b+c-a)] + [abc - b^2(a+c-b)] + [abc - c^2(a+b-c)] \geqslant 0$

But $abc - a^2(b+c-a) = a(bc - ab - ac + a^2)$

$$= a[b(c-a) + a(a-c)]$$

$$= a(a-c)(a-b)$$

LHS

$$a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b)$$

$$(a-b)(a^2 - ac - b^2 + bc)$$

$$(a-b)(a^2 - b^2 - c(a-b))$$

$$(a-b)[(a-b)(a+b) - c(a-b)]$$

$$(a-b)^2(a+b-c) \geqslant 0.$$

QU8 1) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{n}{2(n+2)}$

S1 $n=1$ $\frac{1}{(2)(3)} = \frac{1}{2(3)}$ ✓

S2 Assume true $n=k$ $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{n}{2(n+2)}$

Show true $n=k+1$

$$\sum_{k}^{n+1} \frac{1}{(k+1)(k+2)} = \frac{n+1}{2((n+1)+2)} = \frac{n+1}{2(n+3)}$$

$$\sum_{k}^{n+1} \frac{1}{(k+1)(k+2)} = \sum_{k}^{n} \frac{1}{(k+1)(k+2)} + \frac{1}{((n+1)+1)((n+1)+2)}$$

$$= \frac{n}{2(n+2)} + \frac{1}{(n+2)(n+3)}$$

$$= \frac{n(n+3) + 2}{2(n+2)(n+3)}$$

$$= \frac{n^2+3n+2}{2(n+2)(n+3)}$$

$$= \frac{\cancel{(n+2)}(n+1)}{2\cancel{(n+2)}(n+3)}$$

$$= \frac{n+1}{2(n+3)}$$

S3

Qu 8 ii) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} \frac{n}{2(n+1)}$

from (i)

$$\lim_{n\to\infty} \frac{n}{n}\left(\frac{1}{2(1+1/n)}\right) = 1/2. \qquad S = 1/2$$

iii) $\int_1^\infty \frac{dx}{(x+1)(x+2)} = \int_1^\infty \frac{1}{(x+1)} - \frac{1}{(x+2)}$

(Partial Fraction

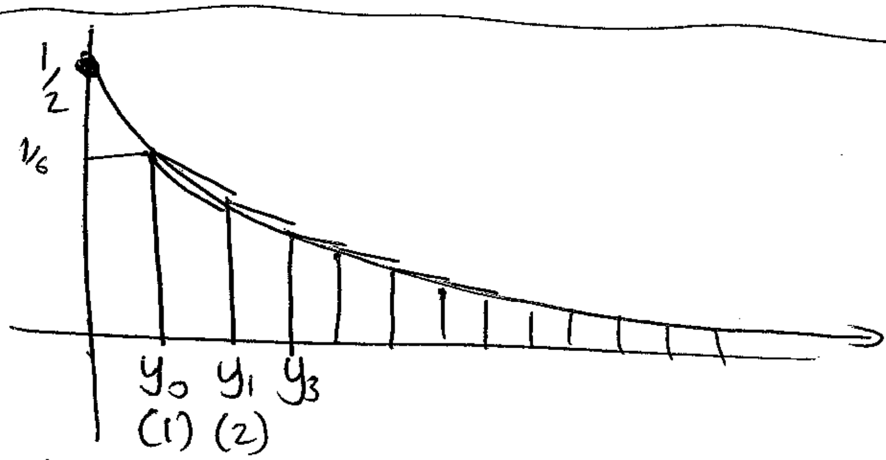
$$= \left[\ln\frac{(x+1)}{(x+2)}\right]_1^\infty$$

$$= \ln 1 - \ln 2/3$$

and $x \to \infty$.

$$= \ln 3/2. \qquad I = \ln 3/2$$

iv(v) $\frac{1}{(x+1)(x+2)}$



$$T = \lim_{n\to\infty} \frac{h}{2}\left(y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots y_{n-1})\right)$$

$h = 1$. $y_0 = 1/6$ $y_n = 0$ {As $n\to\infty$} $2\left(\sum_{k=2}^{n} \frac{1}{(k+1)(k+2)}\right)$

$1/2 - 1/6$

$$T = 1/2\left(1/6 + 0 + 4/6\right)$$

$$T = 5/12.$$

$$I < T < S$$